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EDITED BY

W. H. METZLER

ASSOCIATED WITH

EUGENE R. SMITH HARRY D. GAYLORD

GEO. GAILEY CHAMBERS WILLIAM E. BRECKENRIDGE

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DOES THE STUDY OF MATHEMATICS TRAIN THE MIND SPECIFICALLY OR UNIVERSALLY?

By Ernest C. Moore.

Education is or at least aims to be a conscious process and a purposive undertaking. To teach anything we must first know what purpose is to be served by it and how it must be taught so that that purpose will be served. As there are many subjects which might be studied and many ways in which each one of them might be presented our first and continuing duty is to select from the whole number of possible subjects those few which are indispensable for the purposes of life and when we have done that we must next select from the many possible ways of studying these subjects those few ways of approaching them which are likely to lead to valuable results.

Now, why should one study anything? As nearly as I can discover there are three answers which are given to this question. First, we must study subects because we owe it to them to do so. It is a debt of honor, of reverence, of obeisance, or worship which we should pay them. We do not study them for what they do for us or what they will enable us to do. They are the ends. We are the means. This is subject worship, a kind of liturgical devotion which we are told we must pay to science, literature, mathematics, philosophy when they are hypostatized into self-existing realities. Its favorite call to prayer is science for the sake of science, literature for the sake of literature, knowledge for the sake of knowledge, and art for

art's sake. This is a peculiarly inhuman belief which annually requires the sacrifice of hecatombs of young lives. It seems to us to be just as idolatrous to worship the creations of men's minds as to worship the creations of men's hands. We are recommended to beware of idols. The creator is more to be revered than his creation. When the creation is ascribed virtue in itself, the proper relations are reversed. Knowledge, art, science, literature, philosophy and mathematics exist for man's sake and not he for them. The question always is, what are they to him, what can he make out of them, what can he do with them? Knowledge cannot be its own end. It must be for something. It must perform some work, must offer some assistance, must serve some human purpose. We may take it on credit but the time must come when it wall pay some sort of dividends. If it does not, it is simply useless and unmeaning. It makes no difference in a world in which only such things are regarded as real as make a difference.

The second reason for studying anything is that we cannot get along without it. It is an indispensable aid to us in doing our work. It may serve us in many ways, but we want it because in days to come we shall use it. It is because we are going to read that we study reading, are going to write that we study writing, are going to use geography and history, literature and science as long as we live that we study geography, history, literature and science, and the parts of these studies which are outworn or have no definite utility we omit, giving our attention exclusively to those aspects of them which have abiding value. According to this view studies are for use and education is preparatory. There are so many difficult things that each one of us must know how to do in order to get on with nature and with our fellowmen, that the whole of life is not sufficient for us to learn them. All that we can do in youth is to master the beginnings of a few of the great human operations. Advanced life must help us to perfect our knowledge of them. From this point of view it is immeasurably important that we do not waste our time upon studies or parts of studies which we cannot use in after years and immeasurably important that we study the subjects that have definite utility in such ways that we will go on using them and increasing our mastery of

them through the years that are to come. The school, then, exists to provide special opportunities for us to become acquainted with the first stages of our life business and must introduce us to it in such a way that we shall, from the first, appreciate its meaning and perform it with a growing interest and an expanding sense of its worth, so that when our school days are over we shall know that our education has but begun and will go on applying and using and perfecting our skill in the great arts of which it has taught us the fundamentals as long as we may live. Education, according to this view, is specific throughout. Its purpose is to enable the student to acquire the beginnings of certain indispensable forms of human skill without which he cannot be a society-supporting unit in a world in which men must live and let live and help themselves and each other in doing so. Every form of skill that we attempt to teach him gets its place in the school program solely because he cannot live a civilized life without practicing it. Traditional reasons are not a sufficient warrant for teaching anything. The course of study is to be made with reference to the future, not because of veneration for the past nor because of blind adherence to the prevailing practice of to-day. The training of the young is so serious a responsibility that it must be made throughout a conscious undertaking. Their time must not be wasted and their future must not be trifled away. Nothing must be attempted in their education without demonstrable reasons for attempting it. Few men who have not followed closely the advances which have been made in the science of education in recent years know how completely present-day educational theory differs from the crude traditionalism of an earlier time. The new efficiency program which schools are trying to put into practice now is first to analyze the habits we want the young to form, to set up specific aims by whittling our purposes to the finest point in helping them to form them, and to measure carefully the results which are brought about by instruction. The effort of to-day is to do away with aimless routinary education, by substituting for it an intelligent procedure which shall be as rational as our present knowledge demands and warrants.

The third reason which has been assigned for studying anything is not that we owe it to the thing we are invited to study

to show it this tribute of respect and adoration, or that we shall need it in order to do our part in carrying on the unfinished business of the race. The third reason for studying certain subjects is that they perfect the mind and make it a better mind than it was before. The main province of the school according to this view is to train the mind not by putting it to work upon the matters it will have to work upon as long as it is a living mind, but to prepare it to work upon these matters by working upon others. This might be called indirect education because it maintains that the best way to learn to do one thing is to learn to do another. But if the theory were put as baldly as that, no one would believe it. It is couched in a more seductive form. Certain studies we are told teach us not only to work with their content, but to work with every content. They have far reaching effects, they enable us to do everything we undertake better because we have pursued them. Much of our learning we must get at retail, acquiring it painfully process by process and never getting any more than we bargain for, and mostly less. I have never heard teachers of history, for example, say that studying history teaches anything but history, or teachers of Spanish that studying Spanish teaches anything but Spanish. Just recently we have heard from eminent physical trainers that military training teaches military training and contributes nothing that makes for bodily well-being but much that harms it. But I have heard teachers of Greek and Latin and French and German say that the study of their subject is not intended to teach Greek or Latin or French or German. The study of their subjects is intended to improve the faculties of the mind. They claim to educate by wholesale, to give instruction in preferred subjects. They do not set out to teach their students the subjects which they study; they teach them, they say, something far more valuable. There are many variants of this claim and as nearly as I can discover no one knows exactly what they mean. I heard one man say in a discussion a while ago that he took it as established that we must sharpen an axe on some other material than that which we proposed to cut with it, likening the mind to an axe and the studies which he espoused to a grindstone, but the mind which God gave us is a pretty sharp instrument from the beginning, and we do not need to get inside it

to do any burnishing or repair work there. I find in Professor Keyser's interesting discussion of mathematics some statements which are puzzling and very hard to make out. "The science," he says, "is no catholicon for mental disease. There is no power for transforming mediocrity into genius. It cannot enrich where nature has impoverished. It makes no pretense of creating faculty where none exists, of opening springs in desert minds. . . . The great mathematician, like the great poet or the great administrator, is born. My contention shall be that where the mathematic endowment is found there will usually be found associated with it, as essential implications of it, other endowments in generous measure, and that the appeal of the science is to the whole mind, direct no doubt to the central powers of thought, but indirectly through sympathy of all, rousing, enlarging, developing, emancipating all, so that the faculties of will, of intellect and feeling learn to respond, each in its appropriate order and degree, like the parts of an orchestra to the 'urge and ardor' of its leader and lord." If the study of mathematics can do that or anything like that it is clear that we must all study mathematics, for though many of us have little occasion to use more than the merest elements of this great science, we all want our minds aroused, enlarged, developed and emancipated so that the faculties of will and intellect and feeling will respond. But is Professor Keyser not claiming too much? If mathematics could indeed do these things would it not be the philosophers' stone? And if it can do these things I trust it will not be thought impertinent to ask why it has not done them. Surely no greater harm can be done to any science than to overestimate its claims and mistake its nature and no greater harm can be done to the young than to submit them to a laborious and time-consuming discipline if we are not certain that that discipline can accomplish what we claim that it can accomplish.

Let us stop long enough to understand each other. The question which we are to consider is not the question of the value of mathematics; nobody doubts its value to anyone who has occasion to use it. The quetsion we are to consider is whether it is to be regarded as unlike other studies which are valuable to those who use them and not of much account to those who do not, but is instead a preferred study which is to be pursued not

for the sake of what we can do with it, but for the sake of what it will do to us. The value of mathematics as a tool, a human device for doing its part of the work of the world, is not disputed—it never has been. The value of mathematics as a universal discipline is not proven; it is disputed. Does learning mathematics teach mathematics as Robert Browning said that "learning Greek teaches Greek and nothing else; certainly not common sense if that have failed to precede the teaching"? Or does learning mathematics teach reasoning in general, not to say anything of its power to arouse, enlarge, develop, and emancipate the faculties of will and feeling?

If we go back to the Greeks who invented this great science we find them taking pains to put limits to their reliance upon it: in the "Memorabilia" of Xenophon we are told that Socrates had very decided views as to the value of geometry. "Everyone (he would say) ought to be taught geometry so far, at any rate, as to be able if necessary, to take over or part with a piece of land, or to divide it up or assign a portion for cultivation, and in every case by geometric rule. That amount of geometry was so simple indeed and easy to learn, that it only needed ordinary application of the mind to the method of mensuration, and the student could at once ascertain the size of the piece of land, and with the satisfaction of knowing its measurement depart in peace. But he was unable to approve of the pursuit of geometry up to the point at which it became a study of unintelligible diagrams. What the use of these might be he failed, he said, to see; and yet he was not unversed in these recondite matters himself. These things, he would say, were enough to wear out a man's life and to hinder him from many more useful studies. . . . Socrates inculcated the study of reasoning processes, but in these, equally with the rest, he bade the student beware of vain and idle over-occupation. Up to the limit set by utility he was ready to join in any investigation and to follow out an argument with those who were with him; but there he stopped." [Xenophon: "Memorabilia," IV., 7.]

This passage is thoroughly in keeping with Cleanthes's statement that Socrates cursed as impious "him who first sundered the just from the useful." Socrates's disciple Plato made a larger use of mathematics in the course of study which he out-

lined for the few selected youths whom he proposed to train to be philosopher-kings in the Republic of his vision. You will remember that he prescribed for them a ten years course in arithmetic, geometry, astronomy, and music because these studies lead naturally to reflection, but seem never to have been rightly used. The example which he gives of the way in which he would use these studies shows that he did not rely upon such a knowledge of them as our students are invited to get to lead his disciples to reflection. "When there is some contradiction always present and one is the reverse of one and involves the conception of plurality, then thought begins to be aroused within us and the soul perplexed and wanting to arrive at a decision, asks: 'What is absolute unity?' This is the way in which the study of the one has a power of drawing and converting the mind to the contemplation of true being. You are right, he said; the observation of the unit does certainly possess this property in no common degree, for the same thing presents at the same moment the appearance of one thing and an infinity of things." ("Republic," 524 and 525.) Plato's study of arithmetic is undertaken to consider the nature of numbers, and his geometry, the nature of space. It is intended to lead the student to discover the reality of mind, to know himself the thinker, not the science of mathematics. Will ten years of such study give him a trained mind? These studies, he says, are "useful, that is, if sought after with a view to the beautiful and good; but if pursued in any other spirit, useless. . . . Do you not know that this is only the prelude of the actual strain which we have to learn? For you surely would not regard the skilled mathematician as a dialectician? Assuredly not, he said. I have hardly ever known a mathematician who was capable of reasoning."

We find Aristotle too declaring that "the man of education will seek exactness so far in each subject as the nature of the thing admits, it being plainly much the same absurdity to put up with a mathematician who tries to persuade instead of proving, and to demand strict demonstrative reasoning of a rhetorician. Now each man judges well what he knows and of these things he is a good judge: on each particular matter he is a good judge who has been instructed in it, and in a general way the

man of general cultivation." ("Ethics," 1094*b*.) But this general cultivation is to be gotten by familiarity with many subjects not from the study of any one subject. "The capacity of receiving knowledge is modified by the habits of the recipient mind. For as we have been habituated to learn, do we deem that everything ought to be taught, and the same object presented in an unfamiliar manner, strikes us not only as unlike itself, but from want of custom as comparatively strange and unknown. . . . We ought therefore to be educated to the different modes and amount of evidence which the different objects of our knowledge admit." ["Metaphysics," II., 3.] There is no recognition of mathematics as teaching more than mathematics here. These Greeks do not rely upon it as a training in universal reasoning.

No such claim is made for the study until the faculty psychology brought faculty education in its train some time about the beginning of the eighteenth century. Faculty psychology is everywhere recognized as false doctrine since the criticism of Herbart gave it its deathblow in the early years of the nineteenth century. But faculty education still remains, though the psychologists tell us that there are no faculties to be educated. This of itself is a curious commentary upon the unscientific character of our education; but before I consider the claim that mathematics trains the faculty of reasoning I want to point out that there have from its first appearance as a philosophy of education been almost or quite as many competent critics of this doctrine as upholders of it.

I trust I shall not unduly tax your patience, if I refer to that remarkable article "On the Study of Mathematics as an Exercise of Mind" which Sir William Hamilton published in 1836. Professor Keyser calls it "Sir William Hamilton's famous and terrific diatribe against the science," but opinions of mathematicians seem to differ about it, for Professor Young finds it instructive to the teacher of mathematics and regards it as "a pity that more such criticisms are not made." Whatever else Sir William Hamilton's essay may be, it is not a diatribe against the science of mathematics. He says expressly: "In the ***first*** place that the question does not regard the *value of mathematical* SCIENCE *considered in itself, or in its objective results,*

but the *utility of mathematical* STUDY, that is, *in its subjective effect, as an exercise of mind;* and in the *second,* that the expediency is not disputed, of leaving mathematics as a co-ordinate, to find their level among the other branches of academical instruction. It is only contended that they ought not to be made the principal, far less the exclusive object of academical encouragement. We speak not now of professional but of liberal education; not of that which considers the mind as an instrument for the improvement of science, but of this which considers science as an instrument for the improvement of mind. Of all our intellectual pursuits the study of the mathematical sciences is the one whose utility as an intellectual exercise when carried beyond a moderate extent, has been most peremptorily denied by the greatest number of the most competent judges; and the arguments, on which this opinion is established have hitherto been evaded rather than opposed." If anyone has any doubt about the number of opinions which he musters to support his contention "that the tendency of a too exclusive study of these sciences is absolutely to disqualify the mind for observation and common reasoning" he has only to consult the article to learn how numerous they are. And I do not think it is fair to refute this article by ascribing it to "jealousy, vanity, and parade of learning," or to set it aside by declaring "that Hamilton by studied selections and omissions deliberately and maliciously misrepresented the great authors from whom he quoted . . . d'Alembert, Blaise Pascal, Descartes and others, distorting their express and unmistakable meaning, even to the extent of complete inversion."* It is easy to make charges against men who quote. That is a familiar line of attack. They can be charged with quoting what they should not have quoted, or of not quoting what they should have quoted. Such charges divert attention from what one has quoted but they do not answer it. The question is not whether Sir William Hamilton quotes less than there is to quote—everyone who quotes at all selects what he will quote—and the question is not whether the statement which he quotes in any given case is the average statement of its author upon the subject or the final result of a lifelong consideration of it. These men may have said other things at other

* Keyser, "Mathematics," pp. 23, 24, Columbia University Press. 1907.

times and in other places. They could hardly have been mathematicians without doing so. The question is whether they also at any time or in any place said what Sir William Hamilton quotes them as saying. Did d'Alembert ever say "we shall content ourselves with the remark, that if mathematics (as is asserted with sufficient reason) only make straight the minds which are without a bias, so they only dry up and chill the minds already prepared for this operation by nature." ("Melanges," IV., p. 184, 1763.) It is plain that if he contented himself with that remark we must be contented with that remark as coming from him. And did Descartes say that "the study of mathematics principally exercises the imagination in the consideration of figures and motions" ("Lettres," p. I.–XXX.) and to another correspondent "that part of the mind, to wit, the imagination, which is principally conducive to a skill in mathematics, is of greater detriment than service for metaphysical speculations" ("Epis.," p. II.–XXXIII.) and did Descartes's biographer, Baillet, write "It was now a long time since he had been convinced of the small utility of the mathematics especially when studied on their own account, and not applied to other things. There was nothing in truth which appeared to him more futile than to occupy ourselves with simple numbers and imaginary figures, as if it were proper to confine ourselves to these trifles without carrying our view beyond. There even seemed to him in this something worse than useless. His maxim was that such application insensibly disaccustomed us to the use of our reason and made us run the danger of losing the path which it traces." And does his Life contain the statement that in a letter to Mersenne, written in 1630, M. Descartes recalled to him that he had renounced the study of mathematics for many years; and that he was anxious not to lose any more of his time in the barren operations of geometry and arithmetic, studies which never lead to anything important." And does the author of Descartes's Life in a later passage say "in regard to the rest of mathematics" (he has just been speaking of astronomy) "those who know the rank which he held above all mathematicians, ancient and modern, will agree that he was the man in the world best qualified to judge them. We have observed that after having studied these sciences to the bottom, he had re-

nounced them as of no use for the conduct of life and solace of mankind." ("La Vie de Descartes," I., pp. 111, 112, 225.) It is no answer to such citations to make a great bluster about other statements which might have been quoted and to draw back from these as though it were a profanation even to think of them. The question which must be faced is: Did d'Alembert and Descartes and Descartes's biographer ever at any time say these things? The one legitimate way to attack Sir William Hamilton's use of them as evidence is to deny that they are to be found in the writings of these men. That denial is not made and cannot be made. These are statements which d'Alembert, Descartes and Descartes's biographer made, and made in words which mean exactly what we have indicated and must be reckoned with.

The passage which is quoted from Pascal is quoted at length. In it Pascal says: "There is a great difference between the spirit of mathematics and the spirit of observation. In the former the principles are palpable but remote from common use; so that from want of custom it is not easy to turn our head in that direction; but if it be turned ever so little the principles are seen fully confessed, and it would argue a mind incorrigibly false to reason inconsequently on principles so obtrusive, that it is hardly possible to overlook them. But in the field of observation, the principles are in common use and before the eyes of all. We need not to turn our heads to make any effort whatsoever. Nothing is wanted beyond a good sight; but good it must be, for the principles are so minute and numerous that it is hardly possible but some of them should escape. The omission, however, of a single principle leads to error; it is, therefore, requisite to have a sight of the clearest to discern all the principles; and then a correct intellect to avoid false reasonings on known principles. All mathematicians would thus be observant had they good sight, for they do not reason falsely on the principles they know; and minds of observation would be mathematical could they turn their view toward the unfamiliar principles of mathematics. The cause why certain observant minds are not mathematical is because they are wholly unable to turn themselves toward the principles of mathematics; but the reason why there are mathematicians void of observation is that they

do not see what lies before them, and that accustomed to the clear and palpable principles of mathematics and only to reason after these principles have been well seen and handled they lose themselves in matters of observation where the principles do not allow of being thus treated. These objects are seen with difficulty; nay, are felt rather than seen, and it is with infinite pains that others are made to feel them if they have not already felt them without aid. They are so delicate and numerous that to be felt they require a very fine and a very clear sense. They can also seldom be demonstrated in succession as is done in mathematics, for we are not in possession of their principles, while the very attempt would of itself be endless. The object must be discovered at once by a single glance and not be a course of reasoning, at least up to a certain point. Thus it is rare that mathematicians are observant and that observant minds are mathematical because mathematicians treat matters of observation by rule of mathematics, and make themselves ridiculous by attempting to commence by definitions and by principles a mode of procedure incompatible with this kind of reasoning." ("Pensées de Pascal," p. 1, Article X.)

But Sir William Hamilton is not satisfied with this showing that in learning mathematics we do not learn to reason about all things but only about mathematics; he quotes from scores of other persons to the same effect. His argument is not met by Professor Young's statement, that as mathematics was then taught the subject had, as Sir William Hamilton contended, but small value, "but mathematics is no longer taught as a purely passive subject to-day." That may be true and it is good news if it is true, but Sir William's point is that mathematics cannot be taught in such a way as to enable the student who has studied it, no matter how diligently to reason well about everything. Its lessons have no such universal reference and its methods of reasoning no such universal applicability. The reasoning which life exacts of us is upon contingent matter, the reasoning to which mathematics habituates us is upon necessary matter. In mathematics the premises are given; in life for the most part they must be found. The question we try to answer in mathematics is, what conclusions follow from these premises; the question we are forced to answer in life is, of what principle

is this case an instance or under what principle does this particular belong?

The case against mathematics, not as a science but as a universal trainer of the mind, has become very much stronger since 1830 than it was in Sir William Hamilton's brilliant summary of it. To the crowd of witnesses whom he summoned the names of Huxley and Comte and many another leader of human thought must now be added. The breakdown and abandonment of the faculty psychology left the doctrine of faculty education literally without a leg to stand on. If instead of one memory we have as many memories as the things we remember we cannot train or develop *the* memory for there is none to train. If our nature is so economical that we forget all the things which we have no occasion to remember and remember only those things in which we have taken a lively interest or about which we have built up a net of associations then the way to develop one's memory is to make no effort to develop it, but to spend one's strength instead in finding reasons for being interested in the thing which we want to remember. Let the memory alone, take no memory training lessons, give up forever the notion that a memory ever existed outside of the world of fancy which could remember all things equally well, let the memory alone and give your whole attention to comprehending what you want to remember. That is all that you or anyone else can do. This, you see, requires us to shift our attention wholly from the mind to the content.

The same criticism applies to the training of the reason. No such faculty exists. We reason well about one interest and badly about another. Such a thing as an all round reasoner is not to be found. The agriculturalist reasons well about growing crops, the commission merchant knows more about how to sell them. The geologist reasons well about rocks, the biologist about vital processes, the lawyer about laws, the engineer about the strength of materials, the physician about diseases and the tax expert about the incidents of taxation. The United States wants 150,000 ship carpenters, house carpenters will not do. We are specialists all. The study of mathematics makes a specialist out of the man who pursues it as his life work. How can the same study that makes specialists out of adults make

generalists out of the young? When we study mathematics we learn to make analyses, but to analyze the mathematical "given" is not the same thing nor even the same sort of thing as to resolve an economic situation into its constituent elements or a historical period into the forces which are operating in it or a crime into the factors which indicate its authorship. There are many forms of analysis and only the man who is familiar with a given subject matter can resolve it into its parts. The same thing is true of inferences and of the tracing of relations. The type of analysis or inference which is valid in one field is not valid in another. The universe of facts is no snug-fitting box with interchangeable parts which we can put together and take apart in a few well-defined ways. It is infinitely complex and he who is being trained to operate any part of it must be familiar with the characteristics of his particular field of fact and the processes of manipulation which belong to it. "Going to the root of the matter," says Professor Dewey, in speaking of the doctrine of formal discipline, "the fundamental fallacy of the theory is its dualism; that is to say, its separation of activities and capacities from subject matter. There is no such thing as an ability to see or hear or remember in general; there is only the ability to see or hear or remember something. To talk about training a power mental or physical in general, apart from the subject matter involved in its exercise, is nonsense."

If we turn to the experimental studies which have been made upon this subject we must note that they were not undertaken to inquire whether the memory, or the imaginaion or the observation or the reason can be trained as a faculty. No one who is at all conversant with modern psychology takes that question with any seriousness whatever. Any investigation of it would be a mere waste of time.

Since the psychologists agree that we have a different memory for everything we remember, a different attention for everything to which we attend, a different imagination for everything we imagine, and a different reasoning for everything we reason about, why should there be any investigation to find out to what extent learning to do one thing will help us to do another? The answer is that though our acts are different some of them have common elements and call forth identical responses. If

we learn to count marbles we can count eggs, for the act is the same in both cases, but it does not follow that if we learn to count objects we can count abstractions; that is a new art and must be learned, nor does it follow that if we can count abstractions that we can successfully number objects. There is a great gulf fixed between theoretical and practical arithmetic and between theoretical and practical mathematics throughout. A banker friend of mine declares that counting money in a large bank is so different from counting money in a small bank that city banks hesitate to employ as assistants man who have been trained in country banks. There is much that is common to the two processes but there is at the same time so much that is different that training in one does not prepare for the other.

One who learns to drive a Packard car can drive a Stanley Steamer, that is, he can steer it, for he is only doing over again what he has already learned to do, but one who can adjust a Packard engine cannot adjust the engine in a Stanley Steamer without a special knowledge of that engine.

The ability to use the knowledge which we have acquired in one connection in another is sometimes said to be due to a transfer of training. Professor Dewey tells us that "in the literal sense any transfer is miraculous and impossible." What then does the transfer which is said to take place really mean? Learning to drive a Packard car enables one to drive a Stanley Steamer, because when we drive the Steamer we are simply doing over again what we have already learned to do. Nothing is transferred; instead an act we have already learned to perform is repeated, in a context very like the context in which it was learned. If we could transfer our training from one context to another quite freely we would not go on merely repeating what we have already learned. We would all become inventors. The fact that inventions are and always have been so rare shows quite clearly that we do not do that. We do over and over again what we have already learned to do; but within what limits do we repeat our familiar reactions? That is the question which the experimentalists are answering and their answers all show that the limits go but a little way beyond the lesson itself and that the range of its application is very narrow indeed.

Some of these experiments seek to determine the effects of training in mathematics upon the performance of other kinds of work. One of them is the series of tests conducted by Lewis at Dartmouth. Two test papers were prepared, one containing three originals in geometry, the other three questions in practical reasoning concerning the value of high-school education to the student and the community. Both papers were submitted to 24 different groups of high-school students. The results I give in Mr. Lewis's own words: "If we take the first five mathematical reasoners from each of the 24 groups, we have in all one hundred and twenty pupils most excellent in mathematical reasoning. Of this number 76 or 63 per cent. are at the foot of the practical reasoning series, conspicuous for their inefficiency in practical reasoning. Of the number of pupils at the foot of the mathematical reasoning series, 57, or 47 per cent. are conspicuous for their positions at the head of the practical reasoning series." To supplement this test the records of Dartmouth students in the classes in mathematics and in courses in law were compared. The results were much the same. "Fifty per cent. of the best students in law were conspicuous for their poor showing in mathematics and 42 per cent. of those poorest in law stood at the head of the series in mathematics."

More recently at the University of Illinois Dr. Rugg conducted a classroom experiment in which two groups, one of 413, and the other of 87 college students, were first measured for efficiency in the mental manipulation of spatial elements. The first group of 413 students then took a regular course in descriptive geometry during a college semester of 15 weeks. The other group of 87 college students had no such training during this interval. At the end of the 15 weeks both groups were again measured as they had been at the beginning to discover the effect of the course in descriptive geometry which the one group had taken and the other had not, upon specific abilities in the mental manipulation of spatial elements, (*a*) of a strictly geometrical type; (*b*) of a quasi-geometrical type; and (*c*) of a non-geometrical type. What was the result? Members of both the trained and the untrained group showed improvement in taking the test series a second time. But there were 44 per cent. more gainers in speed in the trained group than in the untrained and

nearly two thirds again as large a proportion of the trained group as of the untrained group gained in accuracy. Of the group that had the training not all gained and of the group that did not have the training a very large number gained as much as those who had had it.

How many individuals gained? "In 'attempts' 67.8 per cent. of the training group and 42.5 per cent. of the control group gain in 60 per cent. or more of the tests taken." That is 42½ folks out of every hundred who did not have the training took the tests as successfully as 68 out of every hundred who did have it. That is, the course seems to have been of some positive assistance in preparing only 25½ folks out of each hundred to take the test. To 32 out of every hundred who took it it was no help and 42½ of every 100 who took it got on just as well without it as with it, that is so far as attempts went. "In 'Rights' 72.7 per cent. of the training group and 31 per cent. of the control group gain in 60 per cent. or more of the tests taken." If 72.7 per cent. who took the training gained we may conclude that 27 out of every hundred who took it did not gain and as 31 per cent. of those who did not have it did as well as those who did have it only 42 out of every hundred became more accurate because of it, while 58 did not, thus you see the chances seem to be about 6 to 4 against expecting anything in the way of general training, that is training which is not strictly specific from such a course. On Dr. Rugg's showing the dice are loaded against every student who takes this course for general training.

It is true, as he points out, that more of those who took the training gained than of those who did not, but a considerable number of those who took it did not gain, and a very considerable number of those who did not take it gained. So to gain it is not necessary to take it and if one does take it there is no certainty that he will gain.

These are his figures but this is not Dr. Rugg's conclusion. His conclusion is that these results supply confirmatory evidence of the "transfer of training," though, as he says, his data do not of course establish conclusively the possibility of transfer. It is not the possibility of transfer but rather the actuality of transfer that concerns educators. His results, like

those of all the experimental studies I have seen, seem to me to assist materially in establishing the fact that we cannot any longer make a philosophy of education out of the doctrine of formal discipline, and they very positively confirm the suspicion with which any such attempt must be met. The burden of proof rests upon those who uphold this theory. It has never been proven, and until it is proven it is mere conjecture wholly insufficient as a theory of instruction.

Education is too serious a business to be allowed to proceed upon chances which are mathematically known to be against the student. Some of those who have investigated the question whether training is transferred declare that it is not. Some affirm that under certain conditions it is sometimes and in some degree; but even when they declare that it is transferred the evidence of transfer is so inconclusive and the amount of the so-called transfer is so slight and the expectation of it so uncertain, that it is the part of wisdom no longer to build houses of learning upon the shifting sands of this doctrine. The investigations have put a cloud upon the title of this theory of education which cannot be removed. It simply does not work. On the solid rock of specific education we can build and must build, for of the results of specific education we can be sure, but as for formal or general discipline, in the words of Professor Spearman "the great assumption upon which education has rested for so many centuries is now at last rendered amenable to experimental corroboration—and it proves to be false."

Harvard University,
Cambridge, Mass.

COLLEGIATE MATHEMATICS IN RELATION TO THE CHANGES PROPOSED IN THE SECONDARY SCHOOL COURSE.*

By Percey F. Smith.

Any discussion as to a reorganization of secondary-school mathematics should be of more than passing interest to teachers of collegiate mathematics. As one looks back and remembers that college teachers exercised a controlling influence on the secondary school curriculum in the respect that the various subjects—algebra, geometry, etc.—were defined by committees dominated by the college element, the interest in the present situation grows.

In the preceding paragraph I refer to the following facts.

1. Present Definitions of the Divisions of Secondary School Mathematics Laid Down Under College Influence. History of the Present Definitions.

At the summer meeting of the American Mathematical Society in 1902, a special committee was appointed by the council of the Society to report on *standard definitions of requirements for admission to colleges and scientific schools,* this committee to act in co-operation with corresponding committees of the Society for the Promotion of Engineering Education, the National Education Association, and other interested bodies. The committee appointed by the American Mathematical Society consisted of Professor H. W. Tyler (chairman), Professors Fiske, Osgood, Young, and Ziwet. The committee reported to the council of the Society on December 29, 1902, and the definitions set by it are those adopted by the College Entrance Examination Board, as specifically stated in the circular issued by the board containing the definitions of the various subjects in which exam-

* Paper read at the winter meeting of the Association at Springfield, March 3, 1917.

inations are set by the Board, which examinations are based upon the requirements thus defined. These definitions standardized by this committee of college teachers have stood unchanged for fourteen years. There can be no doubt from these facts that collegiate authority has dominated the content of the mathematical courses given in secondary schools. Many times I have heard earnest, progressive school men and women express discontent with this situation, and in that position I think them fully justified. If the present movement is due to some extent to a determination to reverse the order of influence, if it is a demonstration on the part of those who know secondary school boys and girls, know their natural difficulties, what they can learn and what they cannot learn, then it seems to me that a reorganization on thorough rational lines is contemplated. Without entering into well-known facts as to the astonishing change in the conditions surrounding the high-school boy and girl of to-day as compared with a decade ago, it should be self-evident to anyone who is looking for results in education that the teachers on the ground ought to and do know how mathematical instruction for which they are responsible should be conducted. Meeting many times with school men to revise examination papers set by the board, I hear constantly the statement that a certain question set by the examiners involves instruction which they have had little or no time to give. It is sufficiently obvious, at least to me, that such a question ought not be on an examination paper, and I heartily support the position of the high-school teacher in this particular. The serious earnest school man knows how much ground he can cover well, and he knows that he is sending to the higher education a better trained boy if he has been taught to do everything thoroughly, even if some of the topics in the definitions are omitted. There is no experience which reacts on a boy as favorably as becoming expert in doing the work set before him. I shall refer to this matter later.

2. Persistence of Errors in Secondary School Texts and Failure to Advance Mathematical Secondary Education by Progressive Teachers Due to Ironclad Character of Content of Courses.

In a reorganization of the secondary-school curriculum omissions should be settled by agreement of school teachers. For example, it must be clear that certain book propositions in geometry should be omitted. If teachers agree on these, their influence should prevent the appearance of these questions on examination papers. I believe that the domination of secondary mathematics by college authority has had a baleful influence upon the progress and evolution of secondary mathematics. Textbooks have become stereotyped, and the progressive author is afraid to write a different book for fear of perpetrating a freak text. The result has been that the same errors are reproduced year after year as authors follow the traditional lines. May I particularize at this point? Take the incommensurable case, that *bête-noir* of the school boy for generations. Few teachers now cover these propositions. But the reason is not, I fear, because the proofs are wrong, but because questions on this case are no longer set on college entrance papers. It is a discouraging thing to see the error inherent in the traditional proofs reproduced year after year in textbooks by authors of standing. The error has been pointed out many times, and it is an easy one to correct.

For example, in the proof of the theorem that two rectangles having equal bases are to each other as their altitudes, the mistake lies in the fact that the proof tacitly assumes that the area-number for a rectangle is proportional to the product of the dimensions, which assumption is later proved as a theorem!

Take another case, the regular polyhedrons, and look at almost any popular text. The discussion begins with the possible cases, and proves that with triangular faces, three or five can meet a vertex, etc. The proposition, however, concerns the *number of possible regular convex polyhedrons.* The argument shows for *a vertex* five possible cases. Why only five possible *polyhedrons?*

The connection is not at all obvious. A conclusion is jumped

at and the act concealed. Euler's theorem on polyhedrons, namely, that the sum of the number of vertices and the number of faces equals the number of edges increased by two, settles the difficulty at once, for this relation is linear, has one solution, and moreover, leads at once to the number of faces in any case.

For example, suppose we take the case when the faces are pentagons. Let f = number of faces. Then obviously,

$$e = \text{number of edges} = 5f \div 2,$$

$$v = \text{number of vertices} = 5f \div 3,$$

$$\therefore \frac{3}{5f} + 2 = \frac{2}{5f} + f, \quad \text{or } f = 12.$$

If the question of the number of regular polyhedrons is discussed, the usual discussion should not be called a proof.

This kind of slipshod logic is doing the school boy perhaps no harm, but the pity is to see the persistence of these errors year after year in otherwise good textbook writing. This "persistence of errors" in textbooks in geometry must be matter of comment to all progressive and alert teachers. It seems that authors fear to depart from the traditional course and exposition given in books that have sold well.

The preceding paragraphs may serve in support of my contention that the dominance of secondary mathematics by college authority has, first, *been harmful in its effect on the content and thoroughness of the teaching*, and second, *has discouraged progressive and alert teachers in experimenting on and developing new ideas with the natural outlet in publication.* We hear constantly that secondary-school teachers are, as a professional class, the most conservative people in the world. There are reasons for this other than the one of which I am speaking, some of them sufficient reasons, but what can we expect of a great body of people teaching year after year the same science, without progress, with no development possible—a dead science, in fact—and because the details of the content of this science were laid down years ago by a committe of college men?

It is because I have a deep sympathy with the progressive teacher that I take a lively interest in the discussion of changes

in the secondary-school program, and it looks to me as if school men are going to rescue secondary mathematics from the lowly condition in which college authority has placed it. If the school men can organize in this reform, can agree and say, "School mathematics is to be reformed without regard to college authority, we are going to teach algebra, geometry and the other subjects as we think these subjects should be taught, and the content of these courses we shall settle for ourselves, and they will be such as will be interesting and vital to the present-day school boy and girl"—if, I say, such a movement should develop, I believe a great service will be done for mathematical science.

What will be the attitude of college teachers in the case? What can it be but satisfied acquiescence and hearty co-operation and support? Is not the inevitable outcome of the reform that ought to be carried through going to be that a boy entering college will know what mathematics he has been taught better than he does now, because he will have been taught fewer things, and especially the hard things will have been mastered by reason of abundant time devoted to them through omission of other topics, and confidence and pride of accomplishment, the most fundamental forces in mental growth and stimulus, will have been fostered and developed? Moreover, with the content of the secondary school courses revised along the *line of human interest*—an easy matter in the case of mathematics—we shall have a different attitude and expectation on the part of college Freshmen toward mathematics. Further, the introduction of much arithmetical work in the high-school course, with insistence upon *speed and accuracy*, will have the result that boys entering college will know how to cipher, and will appreciate that an accurate location of the decimal point is a matter requiring as much care in computation problems as in fixing an employee's salary.

3. Revision of the Definitions of Entrance Requirements Necessary Before the Present Definitions Become Ridiculous.

An inevitable consequence of the present situation and discussion will be an overhauling of the definitions of elementary subjects, and this should not be entrusted to a committee com-

posed of college teachers. The growth of the number of candidates taking the Comprehensive Examination will have a bearing on this question. The line of division between elementary algebra and plane geometry is becoming less clear. Candidates are now examined on elementary mathematics, or advanced mathematics. It is not so evident, however, that any influence is at work in secondary schools tending to break down the divisions between the different topics in advanced mathematics.

If I understand the spirit underlying the present movement for reorganization, then college teachers of mathematics may expect something like the following from the Freshman of the future:

(*a*) He will ***know about*** fewer topics in the various divisions of secondary mathematics.

(*b*) He will know the fundamentals better.

(*c*) He will have confidence in approaching new mathematical subjects, and he will have an interest and eagerness in anticipating such studies.

(*d*) He will expect to be taught new subjects with thoroughness, and will believe that any topic is completed only when completely mastered. He will show by his reaction when difficulties have been overcome by patience on the part of the teacher and industry on his part, that he is conscious of a growth in mental power, and correspondingly interested and enthusiastic.

(*e*) He will expect an immense amount of strict number work in exercises in his new fields of study, for he will infer that his teachers understand that such applications are what he likes and can appreciate. He will devote himself to such applications with enthusiasm, for he will feel that his school work was directed mainly to this end, and he will feel confidence that he can "make good" in such work by virtue of the emphasis which he has himself been taught to place on speed and accuracy in computation.

(*f*) He will expect an informal introduction to new topics of difficulty, and discussion and illustration of new ideas and methods before formal proofs of theorems and formulas are insisted upon by the teacher. When he understands the content

and bearing of a new principle, he will undertake the study of the reasons for it with interest and determination.

(*g*) He will expect much repetition and drill in learning any novel technique, such as is necessary in the calculus, and he will apply himself to master this, because he will feel that it is something worth doing for himself and not because of possible scholarship penalties for failure.

The difference, then, of which the college teacher must take account, is in great part one of the attitude of mind of his students. And he must meet this and satisfy it by giving his pupils what they rightly expect. How will the teacher of collegiate mathematics meet the situation?

First, Will it be Necessary to Merge and Combine Various Divisions of Mathematics now Taught as Separate Subjects?

This question is important and is receiving much attention and thought from college men. Those who take the position that this question must be answered affirmatively advocate a year's course under the title of freshman mathematics, which should include substantially the essentials of college algebra, trigonometry, and analytic geometry. The position of these teachers is partly that of the progressive school men referred to in the above paragraphs, in that they believe that a reorganization of collegiate mathematics is desirable. Personally, I am not aware that this reorganization is actuated by any principle other than a unifying principle, which has for a basis the idea of function. In other words, the motive seems largely a need felt by college teachers for mental satisfaction in organizing various subjects into a logical consistent sequence. My belief is that Freshmen have no mental hunger of this sort to satisfy and will feel no sense of satisfaction while taking the medicine. But I do not wish it understood that I am on principle opposed to fusion. From my standpoint, fusion is of distinctly minor importance as compared with the content and underlying aim of a course, and these should agree substantially with the aims of the reorganized secondary-school course.

A condition which college teachers have to face is the decrease in the minimum requirement in mathematics for entrance

to a college course. It is a fact that 41 per cent. of our colleges and universities now require at most two units in mathematics for admission, and the tendency is to cut down this amount. It may be assumed that the low point of this tendency curve will be reached in the near future, with a following reaction to a normal, sane position. The proposed two-year course in mathematics in the junior high school seems to me a desirable minimum. Then such a reconstruction of the first year of collegiate mathematics ought to be carried out as will make the transition from school to college in the respect of mathematics smooth and continuous. Arithmetical problems will then continue to be those in which the interest can be aroused. In this connection I recall a criticism of a new text on analytic geometry reviewed a few years since by a professor in Harvard University to the effect that the "usual large number of problems of the trivial numerical sort were included by the author."

In the construction of a course in Freshman mathematics it is a matter of the greatest importance that progress in mathematical education should be a ruling principle. A question that arises in my mind in connection with the proposed first year in mathematics in the junior high school is of this nature, namely, *How much progress really has been made in mastering of methods and facts? Is the selection of material wisely done, and has the author looked forward to the future in planning the course?* There is no reason why a course complete in itself—a complete unit, in other words—cannot serve as a stepping stone to the next story in the edifice of mathematical science.

Second, With the Foundations Properly Prepared, What Changes May be Expected in the Teaching of the Calculus by Alert, Progressive Teachers?

The belief commonly accepted years ago that the calculus was inherently a difficult subject and must be reserved for the mature student has happily ceased to exist. In a recent number of the MATHEMATICS TEACHER I find a statement by a well-known educator that the first notions of differentiation and integration will be included in the elective work of the twelfth school year. A comprehensive four-year course in high-school mathematics

could, in fact, properly include simple applications of the calculus. There is in my opinion no work undertaken by the pupil in mathematics in which such a consciousness of gain in power results as from problems in maxima and minima—problems of which the real interest and importance are obvious. Many boys are willing for the first time to say that mathematics is worth something when the endless variety of problems solvable by the simple machinery of the calculus is spread out before them. They are eager to learn to run the machinery.

As I have said above, the college teacher is going to find a different attitude of mind on the part of the future student. This fact is going to influence greatly the conduct of college courses, if the teacher is to be awake to his opportunities. As I look back upon my courses in mathematics when an undergraduate and upon my experience when a young teacher, it seems to me that the spirit underlying the class work in the calculus was something like this:

"Calculus is a difficult subject and to be regarded with awe by the undergraduate. It will be impossible for many of the class to reach even a rudimentary understanding of the subject. Most of you will hardly learn to differentiate or integrate simple forms. Take the textbook and study it. The few who understand it through some unusual power of insight will be marked men—by the teacher and by the class. In other words, some of the class have natural ability in mathematics, and upon this natural ability the teacher relies. All his efforts will be directed to develop this ability in the few who are regarded as exceptions by the class as a whole."

The few students who survived such a position and were regarded as mathematicians would have done as well even under a worse system. But the student of the future who comes through the reconstructed high-school course will put an obligation on the teacher to carry him along with his co-operation to a successful finish. To accomplish this, the teacher will gradually impress him with the power of the calculus in problem work, will show the student what the calculus is for, will not worry him with little points of rigor, a matter for which the healthy boy has no taste, for he judges by results and only these. The critical faculty develops slowly in the schoolboy,

and there will be few students, perhaps none, in any class who will be prompted to doubt an obvious conclusion when connecting details are omitted. The problems taken up will be free of language which the student does not understand and in which he has not the slightest interest. In the effort to satisfy the clamor of engineering teachers for problems in the calculus which will be related to the applications, some authors of recent textbooks have made the serious mistake of crowding the pages with technical language and data which require of the teacher patient and laborious explanation, and these problems fail to arouse the slightest response because remote from the range of the student's experience.

In general terms, the course in calculus will be substantially a problem course, *learning by doing*, with reasons developed when the principle is thoroughly familiar and appreciated. To approach a new principle with ample illustration and explanation, to show the application of it with the sure result of a reaction on the student's part towards the *value* of the principle, is a certain way of arousing his curiosity as to the origin of the truth of the principle. A problem book in the calculus, with help on difficult parts, can be used as a basis for the course, supplemented by a text for purposes of reference.

In such a course reliance should be placed largely upon intuition, for the fundamental truths of the calculus need no demonstration, but merely have to be carefully explained to be believed.

Third, Will not a Supplementary Course following the Calculus Afford a Means of Unifying the Mathematical Training and Serve as Capstone of the Entire Structure?

Practising engineers rarely use the calculus, largely because it is not natural for them to do so, because they are not proficient. Algebra, geometry and trigonometry provide them with mathematical tools. These they learn to use with ease from constant practice in technical studies pursued in college courses. Assume that a general problem course could be offered after the calculus in which a large number of real questions were proposed with all convenient and suitable reference tables at hand—tables of formulas from trigonometry, calculus, numer-

ical tables, slide rules, etc. Let the emphasis be placed upon clear analysis of a problem, the comparative merit of different methods of solution, thorough discussion of results. Let the problems follow in no particular order, be entirely unclassified. Insist upon accuracy in numerical work, rejection of senseless and superfluous digits. Is it not a certainty that such a course would insure in mathematics a proficiency comparable with that resulting from laboratory work in the natural sciences?

To many it will seem that such effects upon collegiate mathematics as are described in the precedign pages may apply only to the distant future. In the article in the MATHEMATICS TEACHER referred to above is printed the following paragraph:

"Whatever may be the future of mathematics, the science will continue to be taught in the secondary schools for many years to come much as it is at present, because of the mere force of inertia if for no other reason. Schools change slowly, and the training of the necessary teachers can proceed only at a certain rate."

I am ready to accept this statement as in agreement with past experience. But I believe school men and women will move in this work of reconstruction with greater speed than tradition leads one to expect. The college teacher will gain largely by their successful efforts.

YALE UNIVERSITY,
NEW HAVEN, CONN.

COMPREHENSIVE EXAMINATIONS.

By Elizabeth B. Cowley.

What test is to be applied in deciding whether the pupil has completed the work of one grade in such a manner that he is qualified to carry the studies of the next? Is it to be decided by the averages of the daily recitations and the weekly or monthly tests which are prepared and marked by the teacher, or is the decision to be based on the result of a final examination set and marked by some one who did not teach the class? Although this question arises in every grade, it is perhaps felt most keenly at the time of transition from secondary school to college. The friends and foes of the certificate system and the old plan examinations have enumerated and defended, or attacked, many vices and virtues in each plan. A not unnatural result of these lively discussions is a desire on the part of some to suggest a compromise in which they hope to retain the strongest points of each and from which they would eliminate the weakest features of both.

Such a compromise is set forth in the so-called "Comprehensive Examinations," which have been adopted at Harvard, Princeton, and Yale, and which are a part of the "New Plan of Admission" to Mount Holyoke, Smith, Vassar, and Wellesley. The essential features are briefly these: (1) the candidate is to be admitted without conditions or not at all; and (2) the evidence submitted is to be of two kinds, (*a*) that offered by the school, consisting of the complete report on four years' work and the principal's estimate of the candidate's qualities, and (*b*) that offered by the student, consisting of comprehensive examinations in four subjects. These examinations are to be those offered by the College Entrance Examination Board. There are, of course, some differences in the method of administration of the system in the different institutions.

The general aspects of the comprehensive plan as a whole have been discussed at length in papers appearing in such maga-

zines as *Education* and the *Educational Review*. The teacher of mathematics is, as a broad-minded educator, interested in the whole scheme. But after he has studied the general question, he will naturally inquire what relations these comprehensive examinations can and may have to the teaching of his own subject. It is the purpose of this paper to offer some suggestions along this line.

The first question may well be: Does mathematics lose anything by the new plan? The teacher who has been accustomed to the old examination system will immediately reply that there is a lowering of standards, because it is only at two of the seven colleges (Princeton and Yale) that every candidate is required to present himself for examination in mathematics. At the other five colleges the student may substitute physics or chemistry. These critics argue that the inevitable result on schools or individuals weak in mathematics will be to slight this subject more than ever and to cover up the deficiencies by offering chemistry or physics on examination. In this connection it is well to draw attention to the fact that the four women's colleges have at their disposal a means of correcting any great abuse along this line, for they require that the candidate's choice of the fourth subject offered on examination must be approved by the committee on admission of the selected college.

Another objection raised is that there is a weakening in the requirement for those who are examined in mathematics, because formerly the candidate had to pass a two-hour examination in plane geometry and at least one (and possibly two) in algebra, while under the new plan there is substituted one three-hour examination. They offer us the following comparisons of the June (1916) papers. Under the old plan a student would receive 67 per cent. in plane geometry by answering four out of six required questions and 71 per cent. in algebra by solving five out of seven problems. But, say the critics, the whole comprehensive paper on elementary mathematics contains only five questions on algebra and four on plane geometry. Hence if this work had been offered on a comprehensive paper it would have been marked 100 per cent., meaning perfect in algebra and geometry. Take it the other way round. A candidate would receive 67 per cent. for answering six of the nine questions on

the comprehensive paper. He could do this by taking two on algebra and four on geometry, or three on each, or four on algebra and two on geometry, or five on algebra and one on geometry. What would each of these combinations have given him under the old plan?—29 per cent. and 67 per cent. for the first combination, 43 per cent. and 50 per cent. for the second, 57 per cent. and 33 per cent. for the third, and 71 per cent. and 17 per cent. for the fourth. But is this a fair statement of actual conditions? Under the old plan a candidate was not required to be prepared on more than one subject at a time and he was given two hours for the examination on each separate subject. Under the new plan he must have all his material in hand at once and he is allowed only one period of three hours to cover all his elementary mathematics. And it must be remembered that a comprehensive paper is not obtained by taking five questions from an old plan paper in algebra and four from another in plane geometry. Nor is it to be forgotten that answer books of comprehensive examinations are to be judged qualitatively as well as quantitatively.

Some teachers see a third loss in the removal of the requirement of a senior review of any mathematics taken in the early years of high school. But have not the colleges been guilty of insisting upon requirements that caused a serious overcrowding in the senior year? All departments must be willing to do their share, if this defect is to be remedied. If we who teach mathematics insist upon our subject and forget the others, are we displaying the reasonableness and sense of proportion which we are fond of numbering amongst the qualities that result from an intensive and extensive study of our subject? There are some persons who think that there is a blessing in disguise in the removal of the requirement of review. Experienced teachers know how perfunctory these reviews sometimes become and they know that, even when they are well conducted, they have sometimes been harmful because too much confidence has been placed in their power to rectify the havoc wrought by the incompetent and ignorant teacher of a beginning class. In considering this question of the senior review it must be kept in mind that the comprehensive examinations must be taken just before the student enters college. This requirement will automatically elimi-

nate a high-school which plans to do superficial work and to complete it early.

What arguments are offered in favor of these comprehensive examinations? It is claimed that the requirement that the one examination shall cover all the mathematics previously studied will discourage one type of candidate who is really not "college material" but who satisfied the letter of the law of the old requirements. This pupil crammed algebra until he could attain a passing grade in that subject and then emptied his mind of that material and proceeded to commit to memory (temporarily) a text in plane geometry with a view to passing in that subject. A candidate who is a mere memorizer will be sadly embarrassed by the comprehensive examination.

Some educators see a distinct gain in the requirement that no student may present himself for examination unless he can satisfy the committee on admission of his selected college that he has actually studied the work upon which he wishes to be examined. This point will appeal to those readers in algebra who have meditated upon the contents of the answer books. They know that too many of the books bear strong internal evidence that the writers have studied nothing beyond quadratics, but have taken their chances on A or A2. Of course, there are some who admire these venturesome young people. Some other persons (who do not teach freshman mathematics in college) assert that a boy who is shrewd enough to pass under these conditions ought to "get by" in college. It is not worth while to argue this matter here. But there is another point worthy of consideration in this connection. Many who venture in this way fail and it must not be forgotten that one of the most widely read of the recent critics of mathematics as a school subject bases some of his assertions upon the percentage of failures in College Entrance Board Examinations in algebra.

Another advantage claimed for the comprehensive examination is that it forces the student to realize that algebra, geometry, and arithmetic are all parts of one big subject, and not separate, isolated, and unconnected studies. Perhaps the weakest point in any system which depends upon several tests based on limited portions of the subject, instead of one final examination on the whole, is the danger that, even though the pupil masters the

details of the various parts, he may fail to get any realization of the relations of the parts to one another. A candidate who contemplates taking the comprehensive examination will be forced, by the mass of material before him, to make an effort to discover some of the fundamental principles and to learn to apply them.

There is another point to which attention may well be directed. Under the new system, where the reader must give not only the more or less mechanical rating already used in marking, but also decide whether certain good and bad qualities are exhibited in each book, there is a splendid opportunity to be on the alert for weakness in arithmetic work. The college teachers of chemistry and physics and the business men tell us that too many college students are lamentably weak "at figures" and are ignorant of some of the simplest operations in arithmetic. We know, alas, that these statements are not made without some foundations of facts. We have before us at the present time an unusual opportunity to attack this arithmetic weakness. But we cannot make the most of the situation unless we can get a positive, definite statement on the examination papers. At the top of the comprehensive paper in English (June, 1916) there is printed: "However accurate in subject matter, no paper will be considered satisfactory if seriously defective in punctuation, spelling, or other essentials of good usage." What we need in mathematics is a statement to the effect that no paper which is seriously defective in arithmetic will be considered satisfactory. We must remember, however, that we shall never get this by merely hoping for it. It may be remarked in passing that the committee on admissions of at least one of the seven colleges mentioned above has sent an urgent request to the College Entrance Examination Board for the insertion of such a statement on the mathematics paper.

Finally, we invite a careful consideration of the system of comprehensive examinations. It depends for its success upon the cordial and intelligent co-operation of all persons concerned; the candidate, his teachers in high school and college, and those who set the examinations and those who read the answer books.

VASSAR COLLEGE,
POUGHKEEPSIE, N. Y.

THE MATHEMATICS OF WARFARE.

How Trigonometry and Calculus are Canned for the Gunner's Convenience.

By J. Malcolm Bird.

The mathematics of warfare is a large subject, with many ramifications; and I shall be able only to identify some of these, and pursue them long enough to indicate the general directions in which they lead. But first I must point out one feature common to almost all war-time mathematics.

Every teacher has at one time or another consoled a student with the remark that while he has plainly made a numerical error, his method is correct. On the field of battle we should have to put it the other end to—that while his method seems correct, he has made a numerical error that renders his result worthless. Perhaps we would also have occasion to observe that he has taken so long to get the answer that even if it were correct the enemy's movements in the meantime would have made it obsolete.

I need not go into argument and example to indicate that the war-time mathematician must usually furnish a prompt answer as well as a correct one. But the field problems of warfare are mainly concerned with the flight of a projectile or the distance and direction of a fixed or moving object. In the one case differential equations are involved, in the other solution of triangles. Now nobody ever lived who could handle differential equations or solve triangles rapidly enough for his results to be of value to a waiting gunner. So most of the mathematics of warfare have to be worked out in advance, and canned for future use in the form of automatic registering instruments and of tables.

I suppose the problems of gunnery are the ones we think of first in connection with the mathematics of warfare. Such problems are peculiarly those of canned mathematics. But in connection with any problem which may come up for solution,

it is inferred that certain initial conditions are known; and before showing you through the canning establishment, I propose to say a few words about the instruments used in obtaining these necessary data.

Powder pressure inside the barrel is measured in an experimental gun, drilled with holes at regular intervals. In each hole is set a hard piston backed by a soft metal plug of known compressibility. The compression of these plugs by a given discharge can then be measured and interpreted in terms of powder pressure in pounds per square inch.

Velocities inside the bore are measured in the same perforated gun. The holes are now fitted with cutter plugs, which project into the bore and are sharpened at the other end. When the passing shell forces the plug up its hole, this sharp end cuts a wire, breaking a circuit and making a spark. The spark passes from a tiny metal point to the smoked edge of a rapidly-rotating disk, burning a hole in the sooty coating. A number of the disks are mounted on the same shaft, and the successive plugs connected up so as to mark successive disks. This does away with the confusion that might result from a lot of marks on the same disk. The average velocity of the shell in each interval of the gun is computed directly from the measured distance between the corresponding spark-marks; we have $D/R = d/r$, with everything but r known.

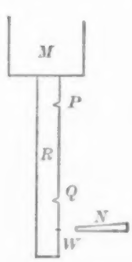

FIG. 1.

Similar in principle is the chronoscope for measuring velocities during flight, illustrated in Fig. 1. The shell here breaks successive circuits by passing through wire screens. In the first circuit is a magnet M, from which hangs a metal rod R; when the circuit is broken the rod falls. The breaking of the second circuit kicks a little sharp-pointed instrument N forward against the falling rod, nicking it at P. The lag is greater in the second circuit than in the first. This is compensated by placing magnet and nicker in a single trial circuit. When this is broken the rod falls and the nicker jumps; and all measurements are made from the nick Q thus obtained rather than from the projection W of the nicker point upon the rod. The arithmetic of the device is, of course, the same as that of the last.

Sometimes we want information about the recoil. The very elegant apparatus for giving us this consists of a tuning fork of known period, mounted to vibrate vertically. A stiff bristle attached to this just brushes against a smoked panel fastened to the side of the gun, and of course paints there a complete picture of the recoil—velocity, acceleration, distance covered.

So much for instruments of measurement. Let us pass next to the canning factory where the relations between the velocities, angles, distances and times of flight are worked out and put up in neat packages for the gun officer to take home. We know, of course, that a projectile fired in a vacuum would trace out a parabola. But our gunners have never mastered the art of shooting in a vacuum; and there might be difficulty in getting the enemy to take up his position in a vacuum to be shot at. So we have to practice gunnery under atmospheric conditions.

This introduces a complication. In a well-ordered universe set up for the special convenience of the mathematician, atmospheric resistance to the passage of a solid body would conceivably be constant—constant with respect to the velocity of the body, with respect to size and weight, with respect to any other conditions which we might impose upon the problem. Unfortunately this is is not the case; resistance varies in one way or another with velocity, with size and weight, and with other factors.

Instead of speaking of the net resistance of the air to the passage of the shell, I shall for the present defer to convention and talk about the shell's ability to make head against that resistance. The one, of course, is numerically the reciprocal of the other. This driving power, or figure of merit of the shell, is called the ballistic coefficient and designated by C. Obviously enough C varies directly as the weight of the shell, inversely as the square of its diameter; but this is only a beginning.

There is for one thing the factor of shape. It is clear that a blunt shell will suffer more from the air's resistance than a sharper one. So we have to strike upon a standard shape, assume that κ, the factor of shape, is unity for that shape, and determine its value for other shapes. The standard shell is taken as one with an ogival head struck on a radius of two diameters. The ogive, I may remark, is that type of arch which

you must all have recognized as characteristic of two things—Gothic architecture, and the modern projectile. It consists of parallel, rectilinear sides, joined by circular arcs of rather long radius. These arcs are tangent to the straight sides, and of course meet in a point to form the arch. The figure shows the shape of the standard "two-caliber" shell head.

Convention demands that we write κ in the denominator of C, so for better shapes than the standard one it will be less than unity. Thus for an ogival head on a radius of four diameters—a four-caliber head, as it is called—κ is .72; for an eight-caliber head it is 0.5. For an absolutely flat head upon a cylindrical shell κ, on the other hand, would be 2.7, while for a similar head, with rounded shoulders, it would be about 2.

Then there is the factor of steadiness, σ, which likewise goes in the denominator of C. A shell that wabbles badly will present periodically a greater resisting surface to the air than one that travels with normal steadiness. Modern gun construction is so well worked out that for all guns and all shells at direct fire σ is practically unity. But any shell wabbles more at high angle fire than we should like to have it, so σ varies with the initial angle. Thus at angles of 45 degrees or more the trajectory has a very sharp vertex. The shell cannot negotiate this turn with complete success, so for the first part of the descent it sets at an angle of some 20 degrees with its own path, gradually settling down into the correct position, one of approximate tangency to the path. Then, too, we read that the Germans have recently found it necessary to put in temporary use guns with badly worn rifling. Under such firing conditions the shell loses part or even all of its spin, so that it is possible for it to fall blunt end down, or actually to capsize and fall sideways, enormously increasing the value of σ. For the low velocity mortars and howitzers ordinarily employed at high-angle fire a fair value of σ for angles exceeding 30° is 1.2.

The proper value of κ is known and constant for a given gun. In determining σ good deal depends upon local and temporary conditions. The factor of tenuity, τ, which must next be fixed, depends entirely upon such conditions. It is established experimentally that the resistance of the air varies directly as its density. So the ballistic coefficient varies inversely as that density.

Moreover, it is found that excellent results are got by assuming the air's density throughout the flight to be equal to its density at the average height of flight. Because of the lower velocity near the vertex, the average height in a parabolic path would be two thirds that of the vertex; and this assumption of a parabolic path, it turns out, leads to no appreciable error in this connection.

Now under standard conditions—barometer 30 inches, temperature 60° F., humidity 67 per cent.—air weighs 534.22 grains per cubic foot. Tables are furnished giving weight of air for certain combinations of barometer and thermometer readings. The gun officer has to look at these instruments, read from his table, usually by interpolation, the weight of air for the observed conditions at the level of the ground, correct the result for the observed humidity, and divide by 534.22. Then he has to get the tenuity at the mean elevation h of his projectile by means of the formula $\tau_h = \tau_g\,(1.00003h)$, or from tables compiled from that formula. The result of all this calculation is τ, and since it is smaller than unity when the air is rarer than the standard, it, too, goes in the denominator of C.

It would seem that we have accounted for everything in the world now, and that there is nothing else to think of. This is not quite the case. Some of the formulæ which we are presently to use involve errors with initial angles greater than 10 degrees. An Italian artillerist has worked out a factor of correction for these errors which is a function of the initial angle. This is called, after its originator, Siacci's Beta-function. For angle of fire 11 degrees it is 1.01, and it increases until for 45 degrees it attains the very considerable value 1.26.

We have now

$$C = \frac{w}{d^2\kappa\sigma\tau\beta};$$

and if R is the total resistance which the air offers to the passage of any shell, and p the resistance to a shell of unit weight and diameter, considered as a function of the velocity alone, we have

$$R = \frac{p}{C}.$$

It is then most natural to ask for an analytic representation of p in terms of v. But, as the old farmer at his first circus said upon beholding the incredible spectacle of the giraffe, "Shucks! They ain't no such animal." All other factors fixed, resistance is indeed a function of velocity, but one that defies all simple analytic representation, one concerning which we can hardly make any definite statement other than that it always increases with v. Its graph (Fig. 2) bears a strong resemblance to the

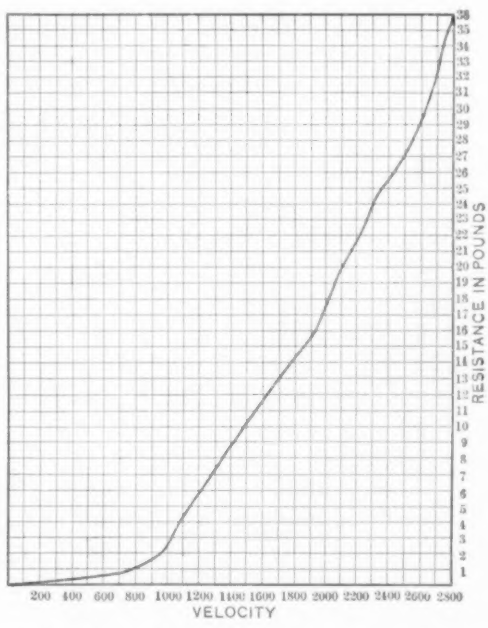

FIG. 2. Curve showing resistance of the air to a projectile 1 inch in diameter.

snakes which we have learned from bitter experience to expect from the weaker students in analytic geometry. Among other eccentricities, it has a tremendous kink in the general neighborhood of the velocity of sound, and another at about 2,413 feet per second—the speed at which air rushes to fill a vacuum. There must be something in this double coincidence, but what or why no man knows. Incidentally, this is as good a place

as any for me to point out that if a shell exceeds this velocity of 2,413 feet per second, it outstrips the efforts of the air to close in behind it, and thus leaves a vacuum in its wake.

Euler, among others, amused himself by deducing what the path of a projectile would be if p varied as the square or the cube of the velocity. Such exercises must now be put in the class of diversions, harmless and useless in equal measure. The determination of the atmospheric resistance, in pounds, to a unit projectile 1 inch in diameter and one pound in weight of the standard shape and under the standard conditions already defined, is wholly empirical. Under these conditions this projectile is fired again and again, and its velocity at various points in its path measured. Comparing the results with those that would obtain if atmospheric resistance were absent, it is easy to deduce the retardation and hence the resistance. In every case, of course, the angle of fire is zero, so that the effect of gravity in diminishing the velocity in the path may be ignored.

The data thus secured have been carefully averaged from countless trials. Together with the differential equations of motion in a resisting medium

$$\frac{d^2x}{dt^2} = -\frac{gp}{C}\cos I; \quad \frac{d^2y}{dt^2} = -\frac{gp}{C}\sin I - g$$

they constitute the raw material from which are constructed the standard ballistic tables that form the chief output of the cannery of war-time mathematics.

These tables are of two sorts—the ballistic tables proper, and the range tables. The former are compiled upon the assumption that we are dealing with direct fire—defined as fire at lower initial angles than 10 degrees. Under these conditions the effect of gravity can be ignored, and the tables constructed almost without reference to the calculus at all.

The first two columns merely set forth the values of p for values of v at intervals of 10 feet per second, as shown upon the graph. Column III. gives ΔT, the time necessary for the projectile to fall from the given velocity to the next one. Since mass $\times$ acceleration $=$ force $\times$ time, we have, of course, $\Delta T = \Delta v/gp$, where Δv is 10 and p is taken as the mean between the two values concerned. Then Column IV. is merely a sum-

mation of the ΔT's to get T, and it works like a definite integral; we find the time it takes the velocity to fall from v_1 to v_2 by subtracting one T-entry from the other. In concise tables for field use the ΔT column is usually omitted, since it consists really of nothing but working figures.

Column V. exhibits $\Delta S = v\Delta T$, the distance traveled while the velocity is dropping Δv feet in ΔT seconds. And of course in column VI. we have S, the summation of the ΔS's: again we subtract the two entries to find the distance traveled in falling from v_1 to v_2, and again we often discard the ΔS column from our finished tables.

The next item involves the effect of g, the gravitational acceleration, in turning the projectile from its initial course. We still work upon the assumption that the initial angle approximates zero; so replacing the cosine by unity and the sine by the angle itself, and making certain other modifications in the differential equations, we get values of ΔD and D, the angle which the axis of the projectile makes with the horizontal plane. This, of course, is the angle I of the differential equations, only here it is measured in degrees, there in radians. As in the previous case we read D by subtraction and discard the ΔD column; and while our results are avowedly derived for low angle fire only, Siacci's Beta-function makes them applicable to all angles.

To complete the ballistic table by the addition of a column showing the height of the projectile above the ground at any point of the trajectory, and to compile range tables which shall exhibit the range for various initial angles and initial velocities, we must fall back upon the differential equations. And here we run into deep water; for in every case we have to deal with an integrand involving both v and p. Now p is a variable, and before integration can be undertaken it must be replaced by its functional value in terms of v. But, as we have seen, this is impossible; so we are at an impassé, and assumptions are in order.

The particular assumptions with which the Gordian knot is cut are rather ingenious. The trajectory is thought of as divided into several partial arcs; and certain of the variables in the intractable integrands are replaced by their mean values

over successive arcs. In this way it is possible to plot the trajectory arc by arc, and to control the error by means of Siacci's Beta-function.

The range tables constructed from the results of such computations consist of double entry rows and columns setting forth the range as a function of initial angle and initial velocity. From the ballistic tables and the general range tables it is customary to compile special tables for use with a particular gun of known muzzle velocity. These exhibit, for angles of elevation no more than five minutes apart, not merely range, but velocity and angle of impact, time of flight, penetration into wrought iron, and various other items of less consequence. Even in the absence of such a table, however, it is possible, since the initial velocity is given and the range is known from the general range table, to refer back to the ballistic tables and pick out the remaining velocity of the projectile at the end of its flight, the time of flight and the angle of fall. The problem is merely that of solving for the lower limit when the upper limit and the value of the definite integral are given.

All general tables, of course, are for unit shell at standard conditions. Since the constant C enters linearly in the differential equations it enters linearly in all results, and has to be used as a modifying factor. The presence of the Beta-function in this constant adjusts what would otherwise be serious error. With the special table giving all data for a particular gun, of course, the factor of correction consists only of τ. All the other factors of C are constant or at worst functions of the initial angle which forms the argument of the table itself; and accordingly these can be considered in compiling the table.

With these tables in his hands the worst thing that can come into the life of the gunner is that he be compelled to select the right table and follow a few cross references through its columns, compute the value of C and multiply. Thus if he is shooting at a moving mark, he must know the time of flight in order to allow the enemy the proper time to catch up with the shell fired in front of him. If, as is universally the case with all heavy artillery in the present war, he is using a time shell instead of a contact one, he must again know time of flight, this time to regulate the length of the fuse. If the enemy is

using, at known range, a gun of known ballistic properties, his engineers will want to know its remaining velocity and angle of fall so as to estimate its effect upon our works, and our engineers will want to know the same things so as to construct those works at the most advantageous angle to secure deflection, and of a thickness to withstand penetration, if this be possible. If the enemy has been located at B on the far side of a hill, and we must know, without making the trial shot that might reveal our position A, whether a shot ranged at angle θ to hit him will

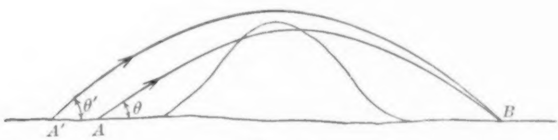

FIG. 3.

clear the hill, or whether we must fall back to A^1 and fire at a higher angle θ^1, we must ascertain the height of our trajectory at the distance of that hill. A wide variety of such problems the gunner, of course, will solve by means of his tables. But in the daily routine of so pointing his gun that it will hit an enemy in a given position, we can, through the agency of range finder and sight, do even better by him.

A range finder is a machine for automatically solving triangles. We set it according to the known parts, and read off on a scale the unknown part in which we are interested—the distance of the object being ranged. While manufactured in a wide variety of models, mathematically range finders fall into just two classes—the horizontal and the vertical types.

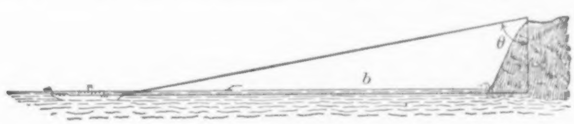

FIG. 4.

The latter is for use in naval work, either in ranging ships from an elevated point on shore, or for use in the crow's nest. Obviously in training a telescope upon a ship from either of these points, a certain angular depression will be necessary.

The telescope, attached to a circular scale, registers this depression automatically; and since the height of the instrument above sea level is known, the triangle is then determined. It is usual to mark the scale with values of tan θ, or if designed for use at a fixed and known height, a, with $a \tan \theta = b$, so that the range can be read directly from the scale.

The horizontal range finder is for more general use. It applies the principle of binocular vision to determine the distance. Our two lines of vision to an object are not parallel; we must look at it cross-eyed; and knowing the distance between our eyes, the angle of ocular adjustment would measure the range.

A layman might object that the contemplated base-line is too small in proportion to the range to give a reliable reading, even with mechanical aid. A mathematician might retort that the astronomer measures star distances of ten light years and more from a base line about seventeen light minutes in length, and that seventeen minutes to ten years is a far smaller ratio than two and one half inches to ten miles. In such an argument the layman, for once, would be right; the ranger has to attain so much greater accuracy than the astronomer that the distance between his eyes is in fact an insufficient base-line for him.

So we have to expand this distance; and the horizontal range finder is the result. Two telescopes, A and A^1, are mounted end to end in a carrying tube C, with their objectives K^1 and K

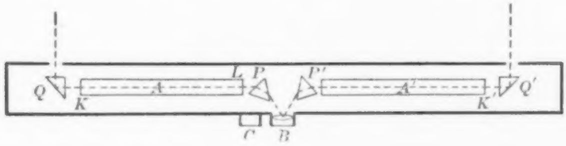

FIG. 5.

in opposite directions. At the eye end L and L^1 of each is a prism P and P^1 which deflects the light passing through both into a common eye-piece B, set in the side of the larger tube. On the opposite side of this tube, at either end, are other prisms Q and Q^1 to deflect into the telescope light falling laterally upon the tube. This light then follows the course of the dotted lines. The observer places his right eye at the common eye-piece of the two telescopes, and of course sees two images of the object

being ranged. By means of a thumb screw he turns the prisms through equal angles until these images coincide exactly. In the meantime his left eye is at another eye-piece *C*, in which is visible a scale and pointer; and this registers the amount of rotation that was necessary to bring the axes of the prisms out of their original parallel position into intersection in the object being ranged. This scale, of course, is not marked in angles, but is calibrated according to the distance between the two apertures at Q and Q^1, so that the operator reads off from it, not angles which have then to be converted into ranges, but ranges themselves. In other words, the range finder, in either of its forms, represents twice-canned trigonometry; the results of triangle solution are canned in the ordinary trigonometric tables, and extracts from these tables, by means of which triangles of a given base, a or QQ^1, may be solved, are canned again on the scale of the range finder. The instrument is then fool-proof; anybody with enough eye-sight and intelligence to know when he is seeing double, should be able to operate it after once being shown how. In some models, it should be remarked, the prisms are replaced by mirrors; but mathematically this does not alter the instrument.

The range known, the artillerist turns at once to his sights. At one time it was actually the practice to look in a book to see what angle of fire to use, and then to set the gun at that angle by means of a protractor. Nowadays we take advantage of the fact that a given gun always shoots the same shell with the same charge, and that initial velocity is therefore a function of the gun, and will not change until we change the gun. We simply attach a vertical sight to the gun, and on it we can the portion of the range tables relating to the initial velocity of that particular gun, entering, of course, on each mark, not the angle, but the corresponding range. When we are given the range as 1,760 yards, it is then only necessary to adjust the hind-sight H so that the 1,760 mark M on it is in horizontal alignment with the fore-sight F. In short guns the fore-sight will be at the muzzle; in long bores this would require too long a hind-sight and too tall a gunner, so the fore-sight must be mounted somewhere along the bore, and the hind-sight, which may be straight or curved, graduated accordingly. But these

are details. The really significant feature of the elevation sight is that when an enemy is visible, rifle or gun is trained upon him by alining the two sights upon him, just as in short range direct fire—and this of course is equally valid when he is above or below the level of the gun with which we are going to blow him off the map. Then when he is invisible we follow the same procedure, using a spirit level to strike a true horizontal line between M and F. So the journeyman gunner doesn't have to learn a new procedure for high angle fire at all. The canned mathematics of the range table is canned a second time for his special benefit upon his sight. It must be correctly canned, however; for if the range be wrong, or the sight be wrong, the sighting may be ever so correct and the shell can only plough up the field a few hundred yards in front of the enemy or sail harmlessly over his head.

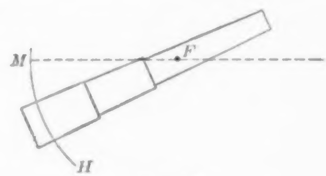

FIG. 6.

When the gun has been properly sighted for range it is still necessary to sight it for direction. Usually the two sights are separate, so that they can be adjusted simultaneously by different men. There can hardly be anything new to the reader in the principle of directing a gun by means of the direction sight; but in its application there enters the interesting factor of drift.

All modern ordnance is rifled. That is to say, the inside of the bore is cut with spiral grooves from end to end, between which project sharp spiral ribs. The body of the shell is of such caliber that it just rests lightly upon these ribs; but the copper rim at the rear has the extreme caliber of the bore, from the bottom of the grooves. As the shell passes up the bore, the sharp, hardened ribs of the rifling cut into this soft copper rim, making grooves in it that have to follow the ribs in their spiral course round and round the inside of the gun. The whole shell consequently acquires an axial rotation as it traverses the bore, and leaves the muzzle with a pronounced spin, which it maintains until it strikes.

This spin adds to the range and makes the flight sufficiently steady to be subject to mathematical laws; it alone renders it

possible for the long, thin projectile to remain head on throughout its flight. This it is that makes essential the use of copper for shell rims, a necessity so embarrassing to Germany at the present time. But the spin produced by the rifling also causes the shell to drift to one side, off its course. One is tempted to describe this offhand as analogous to the curve of a pitched ball, or the spin of a sliced golf ball. In reality, it is a different phenomenon, and one not at all well understood. The baseball or golf ball rotates in a horizontal plane containing the line of

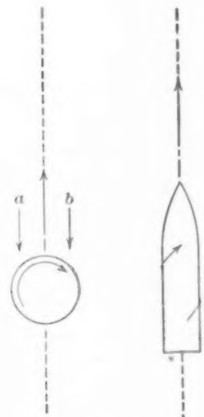

FIG. 7.

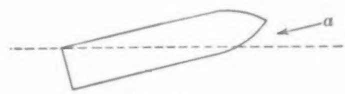

FIG. 8.

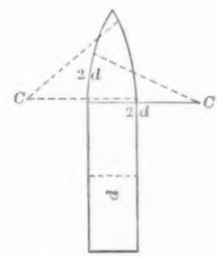

FIG. 9.

flight, while the shell rotates in a vertical plane perpendicular to the line of flight. The slice and the curve ball are explained on the ground of greater pressure on the one side of the advancing face of the ball, due to the slipping off of the air in the direction of rotation. In the case of the shell, rotation is not across the advancing face at all, but around it; so another explanation must be sought.

About a year ago the German technical papers were devoting a good deal of space to this phenomenon, and the following suggestion was brought forth: The ogival head, in following around after the tangent to the path, as it is forced to do by the inertia of the spin, causes a compression of the air beneath the nose in excess of that existing above. The shell always rides at a slight angle to its own path, and there is a sort of denser air cushion

under the advancing head; by virtue of its spin, the shell rolls laterally on this cushion. About that time the German mails ceased to be intermittent and became non-existent, and I have not seen a current German journal in over eight months. Consequently I cannot say how the discussion terminated; but this explanation impressed me as a plausible one. At any rate, the bald fact is that the shell drifts to one side—to the right when shot from a gun with right-handed rifling—and that this deflection increases with the range. Of course, then, after sighting for direction, an arbitrary adjustment of several degrees to the left is necessary.

The possibilities which lie in this complete mathematical control of gun fire reach their climax in the barage fire so extensively employed by our allies. This is a bit of artillery tactics new to the present war. Reduced to its lowest terms, it consists in the training of a large number of guns so that their trajectories all descend along the same straight line. When these guns are then shot at sufficiently short intervals of time, there is a veritable curtain of descending and exploding shells along that line, which cannot be passed by any living thing. The barage may be used to cut the enemy's terrain into two sections, isolated from each other, or to cut him definitely off from a certain portion of our own lines.

Not many years ago this would have been out of the question; the only way to attempt the barage effect would have been to form a single line of identical guns, shooting at the same angle; and it would have been physically out of the question to place them sufficiently close to give an effective barage, to say nothing of the problems of supply, concealment and defense. But today we can distribute guns of all types over an area five or ten miles deep, give each crew the precise range and direction, and go to bed with the secure knowledge that along a given line between us and the enemy, or behind his first defences, there is an impenetrable screen of high explosive shells descending at all angles and all velocities.

It is such artillery co-operation raised to the *n*th power that has made possible the carefully organized operations of our allies in France this spring. An attack is planned several days ahead, and a schedule laid out which must be followed to the

instant by all concerned. For perhaps two days beforehand the objective point in the enemy line is isolated from all supply and all relief by a violent barage descending behind it, and at the same time cut to pieces by intense fire directed at it. When the appointed moment comes, the direct fire ceases, and the barage is shifted to a point between the hostile first lines.

Under cover of this curtain the attackers climb leisurely out of their trenches, form in line, and proceed at a walk across No Man's Land. At scheduled time the barage advances; when necessary the advancing line comes to a halt to wait for the barage to move. The whole journey across No Man's Land is thus screened, save for the last dash of from thirty to fifty yards, after the barage has finally been shifted to a point behind the hostile trenches.

Of course the barage does not afford complete protection. In any event losses are severe in the final rush. But it is plain that this scheme of attack, carefully carried out, makes possible sustained offensive without the excessive losses which characterized the similar operations of the previous years of the war. And this scheme rests wholly upon two factors: the absolute mathematical control of the artillery fire, and the absolute mathematical accuracy with which the women munitions workers of England and France make up the shrapnel shells. If the charging infantrymen could not approach to within a few yards of the descending curtain, with confidence that no undercharged shell is going to fall among them and work havoc, these tactics would be out of the question.

I have almost used up my time, and I have spoken of nothing but gunnery. This is really appropriate, because the problems of observation, which formerly shared with those of gunnery the domain of war-time mathematics, have within the past three years been granted a divorce. Observation is no longer a matter of trigonometry; it has become a matter merely of aviation and photography. The whole region behind the enemy's first line is photographed every day and many times a day. These photographs, taken from a known height and with a known lens, are scaled and charted by geometry too elementary to require description. Nothing can take place that is not at once observed by the aviator, reported, and referred to the proper

spot on these maps. Then, if it is some activity that should be broken up, a few words by telephone to a gun station, giving merely the indexed location on the map, is sufficient to start the breaking-up process.

The aviator, it is true, has a number of ingenious pieces of apparatus to enable him to determine his altitude, the true vertical direction, the amount of sideways drift, etc.; but I pass over these as being hardly within the field of mathematics. Similarly, I refrain from mentioning the mathematics of war-time engineering, since it is in no essential different from the corresponding technique of peace. I believe I have covered the field of distinctively war-time mathematics sufficiently to give a fair idea of what lies therein, so I will go no further. The reader who is sufficiently interested in this subject to wish to pursue it further will find it thoroughly treated in "Modern Guns and Gunnery," by Col. H. A. Bethell, of the British Field Artillery, and the "British Official Text-Book of Gunnery." The former is published by F. J. Cattermole, of Woolwich, and the latter is an official volume gotten out by the British government.

SCIENTIFIC AMERICAN,
233 BROADWAY, NEW YORK CITY.

NEW BOOKS.

The Theory of Evolution. By W. B. Scott. New York: The Macmillan Co. Pp. 183. $1.00.

This is a course of lectures which reviews the evidences on which the theory of evolution is based and the author deserves great praise in presenting the status of this theory among men of science in such a simple way that the general reader may follow it with ease and pleasure. After a general synopsis on the present status of the question the author discusses the evidences from comparative anatomy, from palæontology, from geographical distribution, from experiment, from blood tests and from embryology. It is illustrated and indexed.

A Table of Integrals. By Ralph G. Hudson and Joseph Lipka. New York: John Wiley & Sons. Pp. 25.

A Manual of Mathematics. By Ralph G. Hudson and Joseph Lipka. New York: John Wiley & Sons. Pp. 132.

The tables and formulas here found are taken from the *Engineer's Manual,* and cover mathematics, mechanics, hydraulics, heat and electricity. The smaller work is but a part of the larger.

Projective Geometry. By L. Wayland Dowling. New York: McGraw Hill Book Co. Pp. xiii + 215. $2.00.

This book is the outgrowth of the lectures given by Professor Dowling at the University of Wisconsin. While its primary purpose is to give prospective teachers of mathematics an insight into this field, it is also adapted for use by others.

The method is synthetic, and the text presupposes only elementary geometry and a slight knowledge of trigonometry. The work is well written, the explanations being clear and the book in general well adapted for its purpose.

Essentials of Mechanical Drawing. By L. J. Smith. New York: The Macmillan Co. Pp. vi + 57. 50 cents.

The author of this text evidently believes in the value of mechanical drawing for all pupils. He has written a practical text for a short course designed to give the ability to read ordinary drawings and to make simple sketches. Part of the work does not even require the use of instruments. The book contains many excellent features, and opens an interesting field.

First Course in Algebra. By Herbert E. Hawkes, William A. Luby, and Frank C. Touton. Boston: Ginn and Company. Pp. 301.

This is a revision of the earlier book by the same authors. It seems

to be a better book than its predecessor, and is likely to prove more teachable, though the changes are not revolutionary. One excellent feature is the introduction of numerous oral exercises under the various topics. Lowest common multiple is now treated with fractions, and highest common factor is relegated to the appendix. Many other changes of a minor nature all tend toward simplification without changing the general plan.

The New Barnes Problem Books. By ABRAHAM SMITH. New York: The A. S. Barnes Co. Four books, each of 70 pages. 10 cents each.

The four books are written for the first and second halves of the seventh and eighth grades. They contain an excellent assortment of problems, a large part of them on practical operations, and cover review, new work, and constant practice on fundamentals. These books are worth examining by anyone who needs such list of problems.

How to Make High Pressure Transformers. By F. E. AUSTIN. Hanover, N. H.: Professor F. E. Austin. Pp. 47. 65 cents.

This is the second edition of the book called "Directions for Designing, Making, and Operating High Pressure Transformers." It is written for those who wish to construct their own apparatus for purpose of experiment, and so is especially useful for teachers and students. The book is well written and well illustrated, and should prove valuable to all interested in this line of work.

NOTES AND NEWS.

United States Food Administration, Food Conservation Bureau.

Elementary and Secondary Education.

In planning its campaign the Food Conservation Bureau of the United States Food Administration has realized the importance of the public school as a medium for the dissemination of the ideas which are "to modify the food habits of the one hundred million of our people." It has therefore sought the co-operation of state universities and colleges in order to have the food conservation program reach as large a number of students as possible. A ten-lesson course in conservation was prepared by a committee of domestic science experts, among whom the Department of Agriculture, the Bureau of Education and the United States Food Administration were represented. Every state, except one where there was no summer school, was organized, and co-operation was universally cordial. Six hundred and thirty-three schools received copies of the course, and several hundred thousand students were reached.

In addition to giving the ten-lesson course to summer schools, teachers' institutes were asked to aid in the work. Letters were written to state superintendents, to presidents of state universities and agricultural colleges, and county commissioners, and to each of these a food conservation syllabus was sent. Replies to date have shown enthusiastic co-operation. During the first week there were requests for 28,000 copies of the lessons for institutes held during August, and requests since then have more than doubled that number.

Of the first edition of these lessons, Numbers I.–V., there have been distributed 12,000 copies; of Numbers VI.–X., 10,000 copies. With these have been distributed 145,000 broadsides on food conservation. A new edition of 400,000 copies of Lessons I.–X., inclusive, in one pamphlet, is in press, and orders have already been received for more than half the edition.

With a realization of the enduring need of a conservation program on a broad and fundamental basis the United States Food Administration is planning with the co-operation of the Bureau of Education to place in the schools a course of study which shall be incorporated not as an emergency measure, but as a permanent problem and integral part of our freshened educational aims.

The Bureau of Education will therefore publish, on the first of October and each month thereafter up to June, a bulletin of family and civic economics. The material will be in the form of reading and study courses for elementary and high school grades, and will cover all the topics that enter into community life. These lessons are intended to stimulate closer co-operation between the school and the community in general in solving the problems of our democracy.

Professor Charles H. Judd, dean of the school of education at the University of Chicago, has charge of the preparation of these lessons. Under his supervision, a staff of experienced teachers and educational editors will collect and arrange the necessary material.

North Carolina has effected an organization of the teachers of secondary mathematics for the western portion of the state. The following officers have the work in hand: W. W. Rankin, of the University of North Carolina, president; J. W. Moore, of the Winston-Salem High School, first vice-president; Miss Fannie Starr Mitchell, of the Raleigh High School, second vice-president; L. R. Johnston, of Oak Ridge Institute, third vice-president; J. W. Lasley, Jr., of the University of North Carolina, secretary-treasurer. This organization is the outgrowth of the feeling that the conditions in the secondary schools and colleges of the state would be bettered by a co-operation of the teachers. At the first meeting, held in Greensboro, April 13 and 14, the situation was gone over with care and plans laid which have as their end better teaching of mathematics in North Carolina.

The New York Section of the Association of Teachers of Mathematics in the Middle States and Maryland held its spring

meeting, Friday evening, May 11, at Hunter College, with the following program: Report of Committee on Logarithm Tables, Prof. Elizabeth B. Cowley, Vassar College; Report of Committee on Examinations, Mr. James H. Shipley, High School of Commerce; "The Mathematics of Warfare," by Mr. J. Malcolm Bird, associate editor of the *Scientific American*.

Dr. Edward D. Fitch has resigned his position as treasurer of the association. Dr. Fitch has served faithfully several terms, and has been called from the teaching profession to business life. He will be succeeded by Dr. H. Ross Smith, of the South Philadelphia High School for Boys, who has recently been appointed by the president to fill the unexpired term.

New Members.

Mr. Francis C. Hall, 245 W. 48th St., New York, N. Y.
Nellie H. Bingham, 319 W. 57th St., New York, N. Y.
Mr. A. L. Colston, 1556 73d St., Brooklyn, N. Y.
Mabel E. Reed, 414 Fillmore Ave., East Aurora, N. Y.
Grace E. Barber, 275 West Post Road, White Plains, N. Y.
Isabelle W. Everett, 409 W. 145th St., New York, N. Y.
Alice K. Bielaski, 1 N. Lee St., Cumberland, Md.
Edna Leola Dixon, 2206 Elsinore Ave., Baltimore, Md.
Alice Carey Toon, Mt. Airy, Md.
Forrest A. DeGraff, 455 Ft. Washington Ave., New York, N. Y.
Elizabeth G. White, 2446 Maryland Ave., Baltimore, Md.
Emma K. Root, Stony Point, N. Y.
Lula D. VanDusen, 9 Washington St., Oneonta, N. Y.
Mary L. Pitkin, Sidney, N. Y.
Noah B. Rosenberger, 2d and Chestnut Sts., Coatesville, Pa.
Alvin W. Miller, 3010 Dunbarton Ave., N. W., Washington, D. C.
Blanche I. Loomis, 216 McLennan Ave., Syracuse, N. Y.
Maude A. Barnes, Waverly, N. Y.
Evelyn A. Austin, Sherburne, N. Y.
Gladys T. MacGregory, Hamilton, N. Y.
Bernice Bennett, 16 Academy St., Saranac Lake, N. Y.
George W. Garwaite, 43 Beechwood Ave., Trenton, N. J.